Quaestiones Mathematicae 2015: 1–14.

http://dx.doi.org/10.2989/16073606.2014.??????

This is the final version of the article that is published ahead of the print and online issue

ON SIMPLE PERMANENCE

Dragan S. Djordjević

University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia.
E-Mail dragandjordjevic70@gmail.com

Robin E. Harte

Trinity College, Dublin, Ireland.
E-Mail rharte@maths.tcd.ie

Snežana Č. Živković-Zlatanović

University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, P.O. Box 224, 18000 Niš, Serbia.
E-Mail mladvlad@open.telekom.rs

Abstract. "Simple permanence" is one of several variants of "spectral permanence", which are curiously interrelated.

Mathematics Subject Classification (2010): 46H05, 47A05, 47A53.

Key words: Simply polar elements, semigroup homomorphisms.

0. Introduction. This is a reworking of our previous note [DZH], in which we deployed "Drazin permanence" and *quasipolar* Banach algebra elements in the proof of a variant of the "spectral permanence" enjoyed by C* algebra embeddings. Here we use instead "simple permanence" and *simply polar* elements of semigroups and rings: we believe that the argument is now more transparent and more elementary.

1. Generalized permanence. If $T : A \to B$ is a "semigroup homomorphism" [DZH] then there is inclusion

1.1 $$T(A^{-1}) \subseteq B^{-1} \subseteq B,$$

where A^{-1} is the invertible group of A and hence also

1.2 $$A^{-1} \subseteq T^{-1}B^{-1} \subseteq A;$$

equality here is what is known as the "Gelfand property" or *spectral permanence* for the homomorphism T:

1.3 $$T^{-1}B^{-1} \subseteq A^{-1}\ .$$

[1]This research is supported by the Ministry of Science and Technological Development of Serbia, grant no. 174007.

More generally the "relatively regular" elements

1.4 $$A^\cap = \{a \in A : a \in aAa\}$$

satisfy

1.5 $$A^{-1} = A^{-1}_{left} \cap A^{-1}_{right} \subseteq A^{-1}_{left} \cup A^{-1}_{right} \subseteq A^\cap$$

and if $T : A \to B$ is a semigroup homomorphism then

1.6 $$T(A^\cap) \subseteq B^\cap \subseteq B$$

and hence

1.7 $$A^\cap \subseteq T^{-1}B^\cap \subseteq A.$$

Equality in this case will be described as *generalized permanence* for T:

1.8 $$T^{-1}B^\cap \subseteq A^\cap.$$

We recall [DZH] that spectral permanence does not by itself imply generalized permanence:

Theorem 1. *For ring homomorphisms $T : A \to B$ there is the implication that*

1.9 *spectral and generalized permanence together imply one-one.*

Proof. Generally $T : A \to B$ has spectral permanence only if

1.10 $$T^{-1}(0) \subseteq \mathrm{Rad}(A)$$

has generalized permanence only if

1.11 $$T^{-1}(0) \subseteq A^\cap$$

and evidently

1.12 $$\mathrm{Rad}(A)_\cap A^\cap = O \equiv \{0\}$$

where

1.13 $$\mathrm{Rad}(A) = \{a \in A : 1 - Aa \subseteq A^{-1}\}.$$ □

For a specific example consider the *composition operator*

1.14 $$T = R_\varphi : A = \mathbf{C}^X \to B = \mathbf{C}^Y$$

induced by $\varphi : Y \to X$ where

1.15 $$R_\varphi(a) = a \circ \varphi \ (a \in A).$$

Notice

$$A^\cap = A;\ A^{-1} = \{a \in A : a^{-1}(0) = \emptyset\};$$

of course

$$R_\varphi^{-1}(0) = \{0\} \Longleftrightarrow \varphi(Y) = X.$$

2. Simple polarity. If $a \in A$ has a commuting generalized inverse we shall call it "group invertible" or *simply polar*:

2.1 $$\mathrm{SP}(A) = \{a \in A : a \in a \ \mathrm{comm}(a) a\}.$$

If $T : A \to B$ is a homomorphism then

2.2 $$T\mathrm{SP}(A) \subseteq \mathrm{SP}(B) \subseteq B$$

equivalently

2.3 $$\mathrm{SP}(A) \subseteq T^{-1}\mathrm{SP}(B) \subseteq A.$$

When there is equality here we say that T has *simple permanence*. If we think of the counterimage $T^{-1}B^{-1}$ as in some sense "Fredholm" elements of the semigroup A then the counterimage $T^{-1}\mathrm{SP}(B)$ abstracts what Caradus [C] and Schmoeger [S] have called *generalized Fredholm* operators.

Evidently $a \in A$ is simply polar if and only if it is the commuting product of an invertible and an idempotent; also necessary and sufficient for $a \in A$ to be simply polar is ([X], [HLu]) that

2.4 $$a \in Aa^2 \cap a^2 A \ :$$

recall

$$a^2 u = a = va^2 \Longrightarrow aua = a = ava$$

and take $c = vau$ for a "group inverse". Also necessary and sufficient for $a \in \mathrm{SP}(A)$ in a ring A is ([S]; [KDH] Theorem 5) that there be a "semigroup inverse" $c \in A$ for which

2.5 $$a = aca; \ 1 - ac - ca \in A^{-1}.$$

Notice also

2.6 $$\mathrm{SP}(A) \subseteq A^{\cup} = \{a \in A : a \in aA^{-1}a\} \ :$$

observe that $a + (1 - ac)$ and $cac + (1 - ac)$ are mutually inverse. It follows

2.7 $$\mathrm{SP}(A) \cap A^{-1}_{left} = A^{-1} = \mathrm{SP}(A) \cap A^{-1}_{right} \ :$$

the simply polars are [DHS] "left-right consistent".

THEOREM 2. *If $B^{-1}_{left} \neq B^{-1}_{right}$ there is $T : A \to B$ which is one one and has spectral but not generalized permanence.*

Proof. If $T : A \to B$ and also A is commutative then using (2.7) there is implication

2.8 $$T(a) \in B^{-1}_{left} \setminus B^{-1} \subseteq B^{\cap} \setminus \mathrm{SP}(B) \Longrightarrow a \notin A^{\cap} = \mathrm{SP}(A)$$

violating generalized permanence. In particular if

2.9 $$T = J : A = \mathrm{comm}^2_B(a) \subseteq B$$

then T is one one and has spectral permanence while if $B^{-1}_{left} \neq B^{-1}_{right}$ then we may take $a \in B^{-1}_{left} \setminus B^{-1}$. □

For a specific example ([DZH] Theorem 3.2) take $a = u \in B = L(X)$ or $B = B(X)$ with $X = \mathbf{C}^{\mathbf{N}}$ or $X = \ell_2$ to be the (forward) *unilateral shift*:

$$u(\xi)_{n+1} = \xi_n;\ u(\xi)_1 = 0\ :$$

evidently $vu = 1 \neq uv$ where

$$v(\xi)_n = \xi_{n+1}.$$

Alternatively replace the natural embedding J by the left regular representation L. For another example look at the embedding for a compact Hausdorff space X

2.10 $$C(X) \subseteq \mathbf{C}^X$$

or alternatively for a Banach space X

2.11 $$B(X) \subseteq L(X);$$

here of course spectral permanence follows from the *open mapping theorem.*

THEOREM 3. *When $T : A \to B$ is arbitrary then*

2.12 *T one one with simple permanence $\Longrightarrow T$ has spectral permanence*

while

2.13 *T has spectral and simple permanence $\Longrightarrow T$ one one.*

Proof. The last implication is the argument of Theorem 1; conversely observe

2.14 $$\mathrm{SP}(A)_\cap T^{-1}B^{-1}_{left} \subseteq A^\cup{}_\cap T^{-1}B^{-1}_{left} \subseteq A^{-1} + T^{-1}(0).$$ □

Notice that (2.10) also shows that spectral permanence and one one do not together guarantee simple permanence.

3. Simply polar operators. When $a \in A = L(X)$ is in the ring of additive maps on an abelian group X then necessary and sufficient that $a \in \mathrm{SP}(A)$ is that it is both "of ascent 1" in the sense that

3.1 $$a^{-2}(0) \subseteq a^{-1}(0)$$

equivalently

3.2 $$a^{-1}(0)_\cap a(X) = O$$

and also "of descent 1" in the sense that

3.3 $$a(X) \subseteq a^2(X)$$

equivalently

3.4 $$a^{-1}(0) + a(X) = X.$$

The same conditions characterise simple polarity in the ring of linear mappings on a vector space and also in the ring $A = B(X)$ of bounded linear mappings on a Banach space: here however two or three applications of the open mapping theorem are necessary. For incomplete normed spaces however the conditions (3.1) and (3.3)are not in general sufficient; for example if $a \in A = B(X)$ is one one and onto but not invertible then it will not even be in $A^\cap$. For a specific example take

$$X = c_{00} \subseteq c_0 \subseteq \mathbf{C}^{\mathbf{N}}$$

to be the "terminating sequences" and $a = w \in A$ the *standard weight*

$$x(\xi)_n = (1/n)\xi_n.$$

Together with the assumption $a \in A^\cap$ the conditions (3.1) and (3.3) may still not be sufficient:

Theorem 4. *If $a \in A$ is arbitrary in the ring A then with*

3.5 $$b = \begin{matrix} a & -1 \\ 0 & 0 \end{matrix} \in B = A^{2\times 2} \ \ d = \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix} \in B$$

then automatically

$$b = bdb \in B^\cap$$

while there is implication

$$b \in Bb^2 \Longrightarrow a \in A^{-1}_{left}$$

and also implication

$$b \in b^2B \Longrightarrow a \in A^{-1}_{right}.$$

Hence

$$b \in \mathrm{SP}(B) \Longrightarrow a \in A^{-1}.$$

When in particular $A = L(X)$ is the linear operators on a vector space X then if $a \in A$ is one one or onto then $b \in B$ has ascent or descent one:

$$a^{-1}(0) = O \Longrightarrow b^{-2}(0) = b^{-1}(0) = \frac{1}{a}X$$

and

$$a(X) = X \Longrightarrow b(X^2) = b^2(X^2) = \frac{X}{O}.$$

Proof. Look at the top right hand corner element. □

For example ([H] (7.3.6.8)) we may take again $A = B(X) \subseteq L(X)$ with $X = c_{00} \subseteq c_0$ the space of terminating sequences and $a = w \in A$ the "standard weight". Evidently $a = w$ has a two-sided unbounded inverse in the larger ring $L(X)$ and therefore can have no inverse among the bounded operators $B(X)$.

When $A = B(X)$ for a normed space X and $a \in A$ is of ascent and descent one then [X] each of the following conditions is sufficient for simple polarity:

$$X \ \textit{complete};$$

$$a \in A \ \textit{Fredholm};$$

$$a \in A \ \textit{finite rank};$$

$$b \in X \ \textit{a normed algebra and } a \in \{L_b R_b\} \subseteq B(X).$$

4. Koliha-Drazin permanence. More generally if there is $n \in \mathbf{N}$ for which a^n is simply polar we shall also say that $a \in A$ is "polar" or *Drazin invertible.* If $a \in A$ is polar then there is $c \in A$ for which $ac = ca$ and $a - aca$ is *nilpotent.* More generally still if we write in a Banach algebra A

4.1 $$\mathrm{QN}(A) = \{a \in A : 1 - \mathbf{C}a \subseteq A^{-1}\}$$

for the *quasinilpotents* of A then $a \in \mathrm{QN}(A)$ if and only if $\sigma_A(a) \subseteq \{0\}$ while with some complex analysis we can prove that if $a \in \mathrm{QN}(A)$ then

4.2 $$\|a^n\|^{1/n} \to 0 \ (n \to \infty).$$

Since (4.1) and (4.2) are equivalent it follows that also equivalent ([H2], [K], [HC]) is the condition

4.3 $$\mathrm{QN}(A) = \{a \in A : 1 - \mathrm{comm}(a)a \subseteq A^{-1}\}.$$

In the ultimate generalization of "group invertibility" we shall write $\mathrm{QP}(A)$ for the *quasipolar* elements $a \in A$ those which have a *spectral projection* $q \in A$ for which

4.4 $$q = q^2; \ aq = qa; \ a + q \in A^{-1}; \ aq \in \mathrm{QN}(A).$$

Now [K] the spectral projection and the *Koliha-Drazin inverse*

4.5 $$a^\bullet = q \quad a^\times = (a+q)^{-1}(1-q)$$

are uniquely determined and lie in the double commutant of $a \in A$. It is easy to see that if (4.4) is satisfied then

4.6 $$0 \notin \mathrm{acc}\ \sigma_A(a) \ :$$

the origin cannot be an accumulation point of the spectrum; conversely if (4.6) holds then we can display the spectral projection as a sort of "vector-valued winding number"

4.7 $$a^\bullet = \frac{1}{2\pi i}\oint_0 (z-a)^{-1}dz$$

where we integrate counter clockwise round a small circle γ centre the origin whose *connected hull* $\eta\gamma$ is a disc whose intersection with the spectrum is at most the point $\{0\}$. By the same technique we can display the Koliha-Drazin inverse in the form

4.8 $$a^\times = \frac{1}{2\pi i}\oint_{\sigma'(a)} z^{-1}(z-a)^{-1}dz$$

where $\sigma'(a) = \sigma(a)\setminus\{0\}$. Now generally for a homomorphism $T : A \to B$ there is inclusion

4.9 $$T\ \mathrm{QP}(A) \subseteq \mathrm{QP}(B)$$

while if $T : A \to B$ has spectral permanence in the sense (1.3) then it is clear from (4.6) that there is also "Drazin permanence" in the sense that

4.10 $$\mathrm{QP}(A) = T^{-1}\mathrm{QP}(B) \subseteq A\ :$$

Theorem 5. *For Banach algebra homomorphisms $T : A \to B$ there is implication*

4.11 *spectral permanence $\Longrightarrow$ Drazin permanence.*

Proof. Equality in (1.3) expressed [DZH] in terms of the spectrum together with (4.6). □

Obviously if $a \in \mathrm{SP}(A)$ is simply polar with "commuting generalized inverse $c \in A$ then it is quasipolar and

4.12 $$cac = a^\times\ :$$

its "group inverse" is the same as its "Koliha-Drazin inverse".

As a sort of converse to Theorem 5 and squaring the circle in Theorem 3:

Theorem 6. *If $T : A \to B$ is a Banach algebra homomorphism then*

4.13 $$\mathrm{QP}(A)_\cap T^{-1}(B^{-1}) \subseteq A^{-1} + T^{-1}(0)$$

and if $T : A \to B$ is one one then

4.14 $$\mathrm{QP}(A)_\cap T^{-1}\mathrm{SP}(B) = \mathrm{SP}(A).$$

Hence if $T^{-1}(0) = \{0\}$ is one one then

4.15 *Drazin $\Longrightarrow$ simple $\Longrightarrow$ spectral permanence.*

In particular if $a \in B$ and $T = J : A = \mathrm{comm}^2(a) \subseteq B$ then

4.16 $$A^\cap = T^{-1}\mathrm{SP}(B) = T^{-1}\mathrm{QP}(B).$$

Proof. Uniqueness guarantees that the spectral projection $T(a)^\bullet$ of $Ta \in \mathrm{SP}(B) \subseteq \mathrm{QP}(B)$ commutes with $T(a) \in B$ and one-one-ness guarantees the same for $a \in A$. $\square$

We recall ([DZH] Theorem 3.2) that following (2.8) with $B = B(\ell_2)$ the shift $a = u \in B^\cap \setminus \mathrm{QP}(B)$.

5. Moore-Penrose permanence. By a *star semigroup* we shall understand a semigroup A with an *involution* $* : A \to A$ satisfying for arbitrary $ac \in A$

5.1 $$(a^*)^* = a;\ (ca)^* = a^* c^*;\ 1^* = 1.$$

In rings and algebras involutions are assumed to be additive and "conjugate linear". Obviously there is implication

5.2 $$a \in H(A) \Longrightarrow a^* \in H(A)$$

for each $H(A) \in \{A^{-1} A^\cap \mathrm{SP}(A)\}$. Elements $a \in A$ are said to be *hermitian* or "real" when they are the same as their adjoints:

5.3 $$\mathrm{Re}(A) = \{a \in A : a^* = a\}.$$

A *Moore-Penrose* inverse for $a \in A$ is $c = a^\dagger \in A$ for which the induced idempotents are hermitian:

5.4 $$a = aca;\ c = cac;\ (ca)^* = ca;\ (ac)^* = ac.$$

We write $A^\dagger \subseteq A^\cap$ for those $a \in A$ for which $a^\dagger$ exists. The argument ([HM] Theorem 5) for "C* algebras" works in semigroups [X2] and says that

5.5 $$a^\dagger \in \mathrm{comm}^2(aa^*)$$

is unique and double commutes with $\{aa^*\}$ in A. The "B* condition" in a Banach algebra A says that

5.6 $$\|a^* a\| = \|a\|^2.$$

It follows
$$ax \in A \Longrightarrow \|ax\|^2 \le \|x^*\|\ \|a^* ax\|$$
and hence that $*$ is *cancellable* in the sense that

5.7 $$a \in A \Longrightarrow L_{a^*a}^{-1}(0) \subseteq L_a^{-1}(0);$$

in words ([HLa] Definition 1) the pair $(L_{a^*} L_a)$ is "left skew exact". We need one more object: the "star polars"

5.8 $$\mathrm{SP}^*(A) = \{a \in A : a^* a \in A^\cap\}.$$

Our main objective is to verify again the Harte/Mbekhta observation ([HM] Theorem 6) that in a C* algebra A

5.9 $$A^\cap \subseteq A^\dagger$$

relatively regular elements always have Moore-Penrose inverse and that [HM2] isometric C* algebra homomorphisms have generalized permanence. We begin by collecting some elementary observations:

THEOREM 7. *If the involution $* : A \to A$ is cancellable then there is inclusion*

5.10 $$A^\dagger \subseteq \mathrm{SP}^*(A) \subseteq A^\cap$$

Proof. With cancellation there is implication

$$a \in \mathrm{SP}^*(A) \Longrightarrow a \in aAa^*a \subseteq Aa^*a_\cap aAa$$

and equality

$$\mathrm{Re}(A)_\cap \mathrm{SP}^*(A) = \mathrm{Re}(A)_\cap \mathrm{SP}(A).$$

If $a = aca \in A^\dagger$ with $a^\dagger = c$ then

$$a^*a = a^*(ac)(ac)^*a = a^*acc^*a^*a \in a^*aAa^*a;$$

conversely (5.7)

$$a^*a = a^*ada^*a \Longrightarrow a = ada^*a;$$

hence also

$$a \in Aa^*a \quad \Longleftrightarrow a^* \in a^*aA.$$

Hence if $a^* = a$ then (2.4) follows. □

Now it is clear that isometric C* homomorphisms have "Moore-Penrose permanence":

THEOREM 8. *If $T : A \to B$ is a * homomorphism with simple permanence there is inclusion*

5.11 $$T^{-1}B^\dagger \subseteq A^\dagger.$$

Proof. We claim (cf [K2] Theorem 2.5)

$$A^\dagger = \{a \in A : a^*a \in \mathrm{SP}(A)\}$$

with implication

$$a^*a \in \mathrm{SP}(A) \Longrightarrow a^\dagger = (a^*a)^\times a^*.$$

If $a \in A^\dagger$ with $a = aca$ and $(ca)^* = ca$ and $(ac)^* = ac$ then with $d = cc^*$ we have

$$a^*ad = a^*acc^* = a^*c^*a^*c^* = ca$$

and

$$da^*a = cc^*a^*a = ca.$$

Conversely if $a^*a = a^*ada^*a$ with $a^*ad = da^*a$ and (wlog: $d \mapsto \frac{1}{2}(d + d^*)$) $d = d^*$ then using cancellation with $c = da^*$

$$aca = ada^*a = a \text{ and } ca = da^*a = a^*ad = a^*c^*.$$

Now if $a \in A$ there is implication

$$Ta \in B^\dagger \Longrightarrow T(a^*a) \in \mathrm{SP}(B) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger \ . \qquad \square$$

Thanks to (5.9) this is of course "generalized permanence". The Harte/Mbekhta result is derived by using the "poor man's path" to convert the idempotents ca and ac into self adjoint idempotents. Alternatively thanks to the Gelfand/Naimark/ Segal representation we can look first in the very special algebra $D = B(X)$ of bounded Hilbert space operators:

Theorem 9. *If $d \in D = B(X)$ for a Hilbert space X then*

5.12 $$(d^*d)^{-1}(0) \subseteq d^{-1}(0)$$

and

5.13 $$\mathrm{cl}\ d(X) + d^{*-1}(0) = X;$$

hence

5.14 $$\mathrm{cl}\ d(X) = d(X) \Longrightarrow d^*(X) = d^*d(X) \ \Longrightarrow \mathrm{cl}\ d^*d(X) = d^*d(X).$$

There is inclusion

5.15 $$\mathrm{Re}(D)_\cap D^\cap \subseteq \mathrm{SP}(D);$$

hence

5.16 $$d \in D^\cap \Longrightarrow d \in \mathrm{SP}^*(D) \Longrightarrow d^*d \in \mathrm{SP}(D) \Longrightarrow d \in D^\dagger.$$

Proof. For arbitrary $\xi \in X$ there is [DZH] inequality

$$\|d\xi\|^2 \leq \|\xi\| \ \|d^*d\xi\|$$

and also

$$\mathrm{cl}\ d(X) = d^{*-1}(0)^\perp \ . \qquad \square$$

Both of the Harte/Mbekhta observations now follow:

Theorem 10. *If $T : A \to B$ is isometric then*

5.17 $$T^{-1}(B^\cap) \subseteq A^\dagger.$$

Proof. With $S : B \to D = B(X)$ a GNS mapping we argue using again Theorem 3 together with "spectral permanence at" a^*a (which has of course real spectrum)

$$Ta \in B^\cap \Longrightarrow ST(a^*a) \in \mathrm{SP}(D) \Longrightarrow a^*a \in \mathrm{SP}(A) \Longrightarrow a \in A^\dagger \ . \qquad \square$$

In Theorem 4.2 of [DZH] we established this using the more esoteric $\mathrm{QP}(A)$ rather than $\mathrm{SP}(A)$. It would be entertaining to be able to replace the GNS representation in Theorem 10 with the much more elementary left regular representation $L : A \to B(A)$. Specifically (5.7) enables us to replace $d \in D$ by $L_a \in B(A)$ in (5.12) while if $c = a^\dagger$ is a Moore-Penrose inverse for $a \in A$ then if $a' \in A$

$$c = a^\dagger \Longrightarrow a^*(1 - ac) = 0 \Longrightarrow a' = a(ca') + (1 - ac)a' \in L_a(A) + L_{a^*}^{-1}(0)$$

giving an alternative to (5.13).

6. Polar decomposition. We conclude with a discussion of the "polar decomposition" of C* algebra elements. In the algebra of operators $A = B(X)$ it is familiar that an arbitrary element $a \in A$ can be written as the product of a "partial isometry" and a positive operator. It is not clear that this can be done in a general C* algebra: for example if $A = C[01]$ there are only two idempotents in A and hence only two possible partial isometries. We want here to observe that [H3] at least the Moore-Penrose invertibles have polar decomposition. By a *generalized polar decomposition* for an element $a \in A$ of a C* algebra we shall understand a pair $(uc) \in A^2$ for which $a = uc$ with

6.1 $$u = uu^*u;$$

6.2 $$c = c^*;$$

6.3 $$L_u^{-1}(0) \subseteq L_c^{-1}(0).$$

If in addition

6.4 $$0 \le c \text{ and } L_c^{-1}(0) \subseteq L_u^{-1}(0)$$

then we shall say that (uc) a *polar decomposition* of $a \in A$. We claim ([H3] Theorem 4)

Theorem 11. *If $(uc) \in A^2$ is a generalized polar decomposition of $a \in A$ then*

6.5 $$a^*a = c^2 \text{ and } u^*a = c.$$

If (uc) is a polar decomposition of a then each of u and c are uniquely determined and lie in the double commutant of (aa^*). Also

6.6 $$aa^*u = ua^*a.$$

Proof. For the first part of (6.5) observe that

$$u^*uc - c \in L_u^{-1}(0) \subseteq L_c^{-1}(0);$$

now

$$(u^*a - c)^*(u^*a - c) = c(u^*u - 1)^2 c = 0$$

and the second part of (6.5) follows by cancellation. When (uc) is a polar decomposition then the positivity gives the uniqueness of c:

6.7 $$c = |a| = (a^*a)^{1/2}.$$

The uniqueness of u^*u and uu^* follows from their status as "support" and "cosupport" projections for a; for the uniqueness of u suppose $a = uc = vc$ satisfying (6.1)-(6.4): then

$$(1 - v^*u)c = 0 \implies c(1 - u^*v) = 0 \implies u(1 - u^*v) = 0.$$

Now

$$u^*u = u^*uuv \implies u^*(u - v) = 0$$

similarly $v^*(u - v) = 0$ and hence $v = u$ by cancellation.

It is clear from (6.7) that c is in the double commutant of (aa^*) as are also the support and cosupport u^*u and uu^*. Finally if $d \in \mathrm{comm}(aa^*)$ then it also commutes with each of c u^*u and uu^* and hence

$$cu^*d = dcu^* = cdu^* \implies uu^*d = udu^* \implies duu^* = uu^*d = udu^*$$

and hence

$$du = duu^*udu^*u = uu^*ud = ud.$$

Finally for (6.6)

$$aa^*u = uc^2u^*u = ua^*au^*u = ua^*a\ . \qquad \square$$

We shall write

6.8 $$(uc) = (\mathrm{sgn}(a)|a|).$$

Evidently taking limits of polynomials in a^*a

6.9 $$|a^*|u = u|a|;$$

it follows

6.10 $$(\mathrm{sgn}(a^*)|a^*|) = (\mathrm{sgn}(a)^*\mathrm{sgn}(a)|a|\mathrm{sgn}(a)^*).$$

We can characterise ([H3] Theorem 5) relative regularity in terms of the polar decomposition:

THEOREM 12. *If $a \in A^\dagger \subseteq A$ has a Moore-Penrose inverse then it has a polar decomposition with*

6.11 $$\mathrm{sgn}(a) = (a^\dagger)^*|a|.$$

If $a \in A$ has polar decomposition $(u|a|)$ then

6.12 $$d = |a| + 1 - u^*u \Longrightarrow L_d^{-1}(0) = \{0\}$$

and

6.13 $$a \in A^\dagger \Longrightarrow d \in A^{-1} \Longrightarrow a \in A^\cap.$$

Proof. We argue with $c = a^\dagger$ and $u = c^*|a|$ that

$$uu^*u = c^*|a|^2cc^*a = c^*a^*acc^*|a| = (ac)^*(ac)c^*|a| = c^*a^*c^*|a| = c^*|a|$$

and

$$u|a| = c^*|a|^2 = c^*a^*a = (ac)a = a.$$

If $x \in A$ is arbitrary there is implication

$$dx = 0 \Longrightarrow ucx = 0 \Longrightarrow cx = 0 = u^*ucx = 0 \Longrightarrow u^*ux = 0 = (1 - u^*u)x = 0.$$

Also

$$d \in A^{-1} \Longrightarrow ad^{-1}u^*a = udd^{-1}a^*a = ua^*a = a.$$

Conversely if $a \in A^\dagger$ then $a^\dagger a = u^*u$ and $aa^\dagger = uu^*$ and hence

$$d' = (a^\dagger a + 1 - u^*u) \Longrightarrow dd' = 1 = d'd\ . \qquad \square$$

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Received 29 July, 2013.